$\boxed{10}$ Factorization Structures

<u>Goal</u>: Understan $D(\mathrm{Bun}_G)$

Recall $D(\mathrm{Bun}_G) \hookrightarrow D(\mathrm{Bun}_G^{B\text{-gen}})$

fully faithful

One can show $D(\mathrm{Bun}_G^{B\text{-gen}}) \hookrightarrow D(\mathrm{Bun}_G^{N\text{-gen}})$

<u>Question</u>: Why is $D(\mathrm{Bun}_G^{N\text{-gen}})$ easier given that $\mathrm{Bun}_G^{N\text{-gen}}$ is <u>NOT</u> Artin stack in general

BG is artin:

$$\cdots \; G\times G\times G \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}\; G\times G \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; G \rightarrow 1$$

"colimit of Affine derived schemes w/ smooth morphisms"

<u>Answer</u>:

$$\begin{tikzcd}
\exists\, \mathrm{Whit}(G) \arrow[r, hook] & D(\mathrm{Bun}_G^{N\text{-gen}}) \\
& D(\mathrm{Bun}_G) \arrow[ul, "!"] \arrow[u, hook] \arrow[u, phantom, shift right=6ex, "\circlearrowright"]
\end{tikzcd}$$

factorization structure

Whittaker coefficient

1) Factorization algebras

$Gr_{G,x}$ affine Grassmannian

$x \in C$

In terms of functor of points

$x : S \to C$

$Gr_{G,x}(S)$

$= \left\{ \begin{array}{l} G\text{-bundles on } C_S = C \times S \\ \text{w/ trivialization on } \underbrace{(C \setminus \{x\}) \times S}_{= C \times S \setminus \Gamma_x} \end{array} \right\}$

Prestack

$(Sch^{aff})^{op} \to Spc$

$S \mapsto \mathcal{Y}(S)$

$\uparrow$ test scheme

Recall previously $Gr_G = G(k((t)))/G(k[[t]])$

Define $Gr'_{G,x}$ by:

$Gr'_{G,x} := \left\{ \begin{array}{l} G\text{-bundles on } \mathbb{D}_S \\ \text{w/ trivialization on } \mathbb{D}_S^\times \end{array} \right.$

for $S = \operatorname{Spec} k$

$\mathbb{D} = \operatorname{Spec} k[[t]]$

$\mathbb{D}^\times = \operatorname{Spec} k((t))$

$\mathbb{D}_S := \operatorname{Spec} A[[t]]$

$\mathbb{D}_S^\times := \operatorname{Spec} A((t))$

Thm 1 (Beauville-Laszlo)

$Gr_{G,x} \to Gr'_{G,x}$

is an isomorphism

$$\Rightarrow \quad Gr_{G,x}(k) = Gr_{G,x}(\operatorname{Spec} k) = G(k((t)))/G(k[[t]])$$

$$\begin{tikzcd} Gr_{G,x} \arrow[r] \arrow[d, "\Gamma"] & Gr_{G,C} \arrow[d] \\ x \arrow[r] & C \end{tikzcd}$$

Idea of Beilinson - Drinfeld:

can think about

BD Grassmannian $\to Gr_{G,C^n}$

$$\begin{tikzcd} Gr_{G,C^n} \arrow[d] \\ C^n \end{tikzcd}$$

$C^n(S) = \{(x_1, \cdots x_n) : S \to C\}$

$$Gr_{G,C^n} = \left\{ \begin{array}{l} (x_1, \cdots x_n) : S \to C \\ G\text{-bundle on } C_S \\ \text{w/ triv. on } C_S / \Gamma_x \\ \qquad = \cup \Gamma_{x_i} \end{array} \right\}$$

clear

look at $n=2$

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & Gr_{G,C^2} \arrow[d] \\ (x,y) \arrow[r] & C^2 \end{tikzcd}$$

① if $x=y$, it is $Gr_{G,x}$

② if $x \neq y$, it is $Gr_{G,x} \times Gr_{G,y}$

Heuristic: P_G G-bundle on C w/ triv on $C \setminus \{x,y\}$

$\rightsquigarrow$ P'_G on $C \setminus \{x\}$ $\Leftrightarrow$ P_G on D w triv on $D \setminus \{x\}$

P''_G on $C \setminus \{y\}$ $\quad\Updownarrow$

P_G on $C \setminus \{y\}$ w/ triv on $C \setminus \{x,y\}$

For general $I \xrightarrow{f} J$ surjective map of finite sets

$\Delta_f : C^J \to C^I$

$(C_j)_{j \in J} \to (C_i)_{i \in I}$

(Ran)

$$\begin{tikzcd} Gr_{G,C^J} \arrow[r] \arrow[d] & Gr_{G,C^I} \arrow[d] \\ C^J \arrow[r] & C^I \end{tikzcd}$$

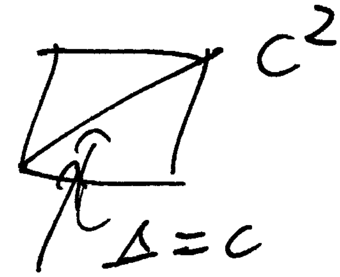
② [Factorization] $\qquad I = I_1 \sqcup I_2$

$$\begin{tikzcd}
(Gr_{G,C^{I_1}} \times Gr_{G,C^{I_2}}) \times_{C^I} (C^{I_1}\times C^{I_2})_{disj} \arrow[r] \arrow[d] & Gr_{G,C^I} \arrow[d] \\
(C^{I_1}\times C^{I_2})_{disj} \arrow[r] & C^I
\end{tikzcd}$$

where $C^{I_1}\times C^{I_2}{}_{disj} = \{ c_i \neq c_j \quad \forall i \in I_1, \ \forall j \in I_2 \}$

Rmk] $Gr_{G,C^I} \to C^I$ formally smooth

ind-scheme of ind-finite type
ind-proper for G reductive



smaller fiber

Defn] D-space over X (D_X-space) is an object of PreStk$/X_{dR}$

$$\begin{tikzcd}
Gr_{G,C^I} \arrow[r] \arrow[d] & Gr_{G,C^I_{dR}} \arrow[d] \\
C^I \arrow[r] & C^I_{dR}
\end{tikzcd}$$

$\leftarrow$ D-space over C^I because

$x_I : S \to C$
$\Leftrightarrow x_I^{red} : S^{red} \to C$

Defn| A <u>factorization</u> <u>space</u> over C is an assignment

$$I \to \mathcal{Y}_I \in \mathrm{PreStk}/C^I_{dR}$$

satisfying the Ran axiom and the factorization axiom

Ex| $Gr_{G, C^I_{dR}}$ is a fact. space

Factorization algebra
~ linearization of factorization space

Defn| A <u>factorization algebra</u> is an assignment

$$I \to A_{C^I} \in D(C^I) = QC(C^I_{dR})$$

s.t. ① $\forall I \overset{f}{\twoheadrightarrow} J$ $\quad \Delta f : C^J \to C^I$

$$\Delta_f^! A_{C^I} \simeq A_{C^J}$$

② (factorization)

$$A_{C^I}\big|_{(C^{I_1} \times C^{I_2})_{disj}} = \left(A_{C^{I_1}} \boxtimes A_{C^{I_2}}\right)\Big|_{(C^{I_1} \times C^{I_2})_{disj}}$$

for $I = I_1 \amalg I_2$

$\mathcal{F} \in D(X)$ $\quad \mathcal{G} \in D(Y)$

$\mathcal{F} \boxtimes \mathcal{G} := \pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{G}$

where $\pi_1 : X \times Y \to X$

$\pi_2 : X \times Y \to Y$

Ex| $I \mapsto \omega_{C^I}$ is a factorization algebra

[Recall $\mathcal{X} \xrightarrow{p_{\mathcal{X}}} pt$, $\omega_{\mathcal{X}} := p_{\mathcal{X}}^! k$]

①
$$\begin{tikzcd} C^J \arrow[rr, "\Delta_f"] \arrow[dr] \arrow[rr, phantom, shift right=3, "\circlearrowleft"] & & C^I \arrow[dl] \\ & pt & \end{tikzcd}$$

② $\omega_{C^{I_1}} \otimes \omega_{C^{I_2}} = \omega_{C^I}$

More generally, given a factorization space $\{\mathcal{Y}_I\}$ over C, one can construct a fact. algebra $\mathcal{A}_{C^I} := \pi_{I,dR,*}\, \omega_{\mathcal{Y}_I}$ ← Borel-Moore homology

where $\pi_I : \mathcal{Y}_I \to C^I$

provided $\mathcal{Y}_I$ is nice enough

e.g. $\mathcal{Y}_I$ is ind-scheme ~~of~~ ind-finite type

Summary

fact $A(x_1 \cdots x_n) \simeq A_{x_1} \otimes \cdots \otimes A_{x_n}$

if $x_i \neq x_j$ $\forall i,j$

Ran $\Rightarrow$ A depends only on $\{x_1 \cdots x_n\} \subset C$ subset

all information is in $\mathcal{A}_X \leftarrow I = \{pt\}$
together with collision data

<u>Big picture</u> (Interlude)

$$D(\mathrm{Bun}_G^{N\text{-gen}})$$

$$\rotatebox{90}{$\rightsquigarrow$}$$

$$\mathrm{Fun}_{\mathbb{Q}}\big(\cancel{N(K)}\backslash G(\mathbb{A})/G(\mathbb{O})\big)$$

$K = k(C)$
function field

$\{\mathrm{Gr}_{G,C^I}\}_I \rightsquigarrow G(\mathbb{A})/G(\mathbb{O})$

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2) Group actions on Categories

i) Sheaves of Categories (?)

$\mathrm{ShvCat}/\mathcal{Y}$ for a prestack $\mathcal{Y}$

<u>Goal:</u> define this

$S = \mathrm{Spec}\, A$ affine derived scheme

$\mathrm{ShvCat}/S = QC(S)\text{-mod}(\mathrm{DGCat})$

$QC(S) = (A\text{-mod}, \otimes)$

is a comm alg obj. in DGCat

<u>classical</u>: $A \in Alg = Alg(Vect) \leftarrow \exists \mu : A \otimes A \to A$

$M \in A\text{-mod} = A\text{-mod}(Vect) \leftarrow A \otimes M \to M$

<u>now</u>: $A\text{-mod} \in Alg(DGCat) \leftarrow A\text{-mod} \otimes A\text{-mod} \to A\text{-mod}$

$\mathcal{F} \in (A\text{-mod})\text{-mod}(DGCat) \quad A\text{-mod} \otimes \mathcal{F} \to \mathcal{F}$

$\mathcal{F} \in (A\text{-mod})\text{-mod}(DGCat)$

$$\Leftrightarrow A \to HC(\mathcal{F}) = End(id : \mathcal{F} \to \mathcal{F})$$

<u>Def</u> $ShvCat/Y = \lim_{S \to Y} ShvCat/S$

$S \to T \rightsquigarrow f^* \; ShvCat/T \to ShvCat/S$

$\mathcal{C} \longmapsto QC(S) \otimes_{QC(T)} \mathcal{C}$

$QC(Y)\text{-mod}$

$\Gamma : ShvCat/Y \longrightarrow DGCat$

$\mathcal{C} \longmapsto \Gamma(Y, \mathcal{C}) = \lim_{S \xrightarrow{f} Y} \Gamma(S, f^* \mathcal{C})$

$\Gamma : QC(Y) \longrightarrow Vect$

$\mathcal{F} \longmapsto \Gamma(Y, \mathcal{F}) = \lim_{S \xrightarrow{f} Y} \Gamma(S, f^* \mathcal{F})$

$\to \mathcal{O}(Y)\text{-mod}$

$$\mathrm{ShvCat}/\mathcal{Y} \xrightarrow{\sim} \mathrm{DG\ Cat}$$

$$\begin{array}{ccl} \cup & & \\ QC/_\mathcal{Y} & \Leftrightarrow & QC(S) \\ & & \in QC(S)\text{-mod}(\mathrm{DG\ Cat}) \\ & \longrightarrow & \{QC(S)\} \otimes QC(\mathcal{Y}) \end{array}$$

$$\mathcal{O}_\mathcal{Y} \Leftrightarrow \{\mathcal{O}_S\}_{S \to \mathcal{Y}}$$
$$\mapsto \mathcal{O}_\mathcal{Y}$$

Recall $\mathcal{Y}$ is called <u>0-affine</u> if $\Gamma : QC(\mathcal{Y}) \to \Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y})\text{-mod}$ is an equivalence

<u>Defn</u> (Gaitsgory) $\mathcal{Y}$ is called <u>1-affine</u> if $\Gamma : QC/\mathcal{Y} \to QC(\mathcal{Y})\text{-mod}(\mathrm{DG\text{-}cat})$ is an equivalence

<u>Ex</u>
- quasi-separated quasi-compact schemes
- Artin stacks of almost finite type
- For S of finite type, S_{dR}

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non-example:
- $\mathbb{A}^\infty = \varinjlim \mathbb{A}^n$

Defn $G\text{-cat} := \mathrm{ShvCat}/\mathbb{B}G_{dR}$

why? $G\text{-rep} = \mathrm{Rep}\,G = QC(\mathbb{B}G)$

$(\rho, V) \longmapsto V$ underlying vector space

$\bullet \circlearrowleft G$

$G \circlearrowright_i \bullet \quad G \circlearrowright V$

$\longmapsto V^G$ invariants

$\mathbb{B}G \xrightarrow{p} \{pt\}$

$\Gamma : QC(\mathbb{B}G) \to \mathrm{Vect}$

$i : (\rho, V) \longmapsto \Gamma(\mathbb{B}G, V) = V^G$

$\pi : pt \to pt/G$

$QC(\mathbb{B}G) \xrightarrow{\pi^*} \mathrm{Vect}$

$ii : (\rho, V) \to V$

$$\mathrm{ShvCat}/\mathbb{B}G_{dR} \to DG\,\mathrm{Cat}$$

$$\mathcal{C} \longmapsto \Gamma(pt, \pi^*\mathcal{C}) = \mathcal{C}^G$$

$$\Gamma \quad \mathrm{ShvCat}/\mathbb{B}G_{dR} \to DG\,\mathrm{Cat}$$

$$\mathcal{C} \longmapsto \Gamma(\mathbb{B}G_{dR}, \mathcal{C}) = \mathcal{C}^G$$

$\mathbb{B}G_{dR}$ is not 1-affine